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On the Stability of the Finite-Difference Time-Domain Method[1]

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In this paper we give a necessary and sufficient condition for the stability of the finite-difference time-domain method (FDTD method). This is an explicit time stepping method that is used for solving transient electromagnetic field problems. A necessary (but not a sufficient) condition for its stability is usually obtained by requiring that discrete Fourier modes, defined on the FDTD grid, remain bounded as time stepping proceeds. Here we follow a different approach. We rewrite the basic FDTD equations in terms of an iteration matrix and study the eigenvalue problem for this matrix. From the analysis a necessary and sufficient condition for stability of the FDTD method follows. Moreover, we show that for a particular time step the 2-norm of the FDTD iteration matrix is equal to the golden ratio.

Key Words: finite-difference time-domain method; stability condition.

1. INTRODUCTION

The finite-difference time-domain method (FDTD method) is widely used for solving transient electromagnetic field problems. As an illustration, at the URL address `http://www.fdtd.org/` a list of over 3400 publications about the FDTD method and its applications can be found. The method is an explicit time stepping method and is based on the classical Yee scheme. A necessary (but not a sufficient) condition for its stability is usually obtained by requiring that the amplitude of a discrete Fourier mode remains bounded as time stepping proceeds (see, for example, Taflove [3] and Kunz and Luebbers [2]).

In this paper we follow a different approach. First, we rewrite the basic FDTD equations in a form which shows that one step of the FDTD method can be interpreted as a forward–backward substitution. Second, we rewrite this form in terms of an iteration matrix. The stability analysis of the FDTD method then amounts to an eigenvalue analysis of this

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iteration matrix. From the analysis a necessary and sufficient condition for stability follows. Moreover, we show that the 2-norm of the FDTD iteration matrix is equal to the golden ratio for a suitably chosen time step.

To specify position, we employ the vector $\boldsymbol{x}$ with Cartesian coordinates $x_1,\ x_2$, and x_3. Further, $\partial_1,\ \partial_2$, and ∂_3 denote differentiation with respect to $x_1,\ x_2$, and x_3, respectively, while ∂_t denotes differentiation with respect to the time coordinate t.

The paper is organized as follows. In Section 2, we introduce Maxwell's equations. These equations are normalized and are written in a particular matrix operator form. This form serves as a basis for our analysis. In Section 3, we briefly review the basic FDTD equations. Everything in this section is well known, only the notation is different. In Section 4, we present our stability analysis, and in Section 5, we show how to apply the stability condition in practice. Finally, in Section 6, we illustrate our results for a simple one-dimensional configuration.

2. MAXWELL'S EQUATIONS

Maxwell's equations describing the behavior of the electromagnetic field in an inhomogeneous, isotropic, and lossless medium are given by

$$-\boldsymbol{\nabla} \times \boldsymbol{H} + \varepsilon \partial_t \boldsymbol{E} = -\boldsymbol{J}^{\mathrm{e}}, \tag{1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \mu \partial_t \boldsymbol{H} = -\boldsymbol{K}^{\mathrm{e}}. \tag{2}$$

In these equations, $\boldsymbol{E}$ is the electric field strength (V/m), $\boldsymbol{H}$ is the magnetic field strength (A/m), ε is the permittivity (F/m), μ is the permeability (H/m), $\boldsymbol{J}^{\mathrm{e}}$ is the external electric-current density (A/m^2), and $\boldsymbol{K}^{\mathrm{e}}$ is the external magnetic-current density (V/m^2). Before we review the basic FDTD equations, we first normalize Maxwell's equations and discuss some of the structure of these equations.

Given a problem-related reference length L, we write

$$\boldsymbol{x}' = L^{-1}\boldsymbol{x} \quad \text{and} \quad t' = c_0 L^{-1} t, \tag{3}$$

where c_0 is the electromagnetic wave speed in vacuum. Further, we introduce the normalized quantities

$$\boldsymbol{E}'(x', t') = \boldsymbol{E}\left(L\boldsymbol{x}', c_0^{-1}Lt'\right), \quad \boldsymbol{H}'(\boldsymbol{x}', t') = Z_0\boldsymbol{H}\left(L\boldsymbol{x}', c_0^{-1}Lt'\right), \tag{4}$$

and

$$\boldsymbol{J}^{\mathrm{e}\prime}(\boldsymbol{x}', t') = LZ_0\boldsymbol{J}^{\mathrm{e}}\left(L\boldsymbol{x}', c_0^{-1}Lt'\right), \quad \boldsymbol{K}^{\mathrm{e}\prime}(\boldsymbol{x}', t') = L\boldsymbol{K}^{e}\left(L\boldsymbol{x}', c_0^{-1}Lt'\right), \tag{5}$$

in which $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$. These quantities satisfy the equations

$$-\boldsymbol{\nabla}' \times \boldsymbol{H}' + \varepsilon_{\mathrm{r}}\partial_{t'}\boldsymbol{E}' = -\boldsymbol{J}^{\mathrm{e}\prime}, \tag{6}$$

$$\boldsymbol{\nabla}' \times \boldsymbol{E}' + \mu_{\mathrm{r}}\partial_{t'}\boldsymbol{H}' = -\boldsymbol{K}^{\mathrm{e}\prime}, \tag{7}$$

where $\boldsymbol{\nabla}'$ is the nabla operator with respect to the primed spatial coordinates, and $\partial_{t'}$ denotes differentiation with respect to t'. Further, $\varepsilon_{\mathrm{r}} = \varepsilon/\varepsilon_0$ and $\mu_{\mathrm{r}} = \mu/\mu_0$ are the relative permittivity and permeability, respectively. In what follows, we drop the primes.

Written out in full, Eqs. (6) and (7) can be arranged in the form

$$(\mathcal{S} + \mathcal{M}\partial_t)\mathcal{F} = \mathcal{Q}, \tag{8}$$

where $\mathcal{F} = \mathcal{F}(\boldsymbol{x}, t)$ is the field vector consisting of the components of the electric field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{H}$ as

$$\mathcal{F} = [E_1, E_2, E_3, H_1, H_2, H_3]^T, \tag{9}$$

and $\mathcal{Q} = \mathcal{Q}(\boldsymbol{x}, t)$ is the source vector composed of the components of the external electric-current source $\boldsymbol{J}^{\mathrm{e}}$ and the components of the external magnetic-current source K^{e} as

$$\mathcal{Q} = -\left[J_1^{\mathrm{e}}, J_2^{\mathrm{e}}, J_3^{\mathrm{e}}, K_1^{\mathrm{e}}, K_2^{\mathrm{e}}, K_3^{\mathrm{e}}\right]^T. \tag{10}$$

The time-independent medium matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \mathrm{diag}(\varepsilon_{\mathrm{r}}, \varepsilon_{\mathrm{r}}, \varepsilon_{\mathrm{r}}, \mu_{\mathrm{r}}, \mu_{\mathrm{r}}, \mu_{\mathrm{r}}). \tag{11}$$

This matrix is positive definite since the relative permittivity and permeability are always positive.

The spatial derivatives are contained in the spatial differentiation operator matrix $\mathcal{S}$ given by

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & \partial_3 & -\partial_2 \\ 0 & 0 & 0 & -\partial_3 & 0 & \partial_1 \\ 0 & 0 & 0 & \partial_2 & -\partial_1 & 0 \\ 0 & -\partial_3 & \partial_2 & 0 & 0 & 0 \\ \partial_3 & 0 & -\partial_1 & 0 & 0 & 0 \\ -\partial_2 & \partial_1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

We also introduce the diagonal matrices δ^{E} and δ^{H} as

$$\delta^{\mathrm{E}} = \mathrm{diag}(1, 1, 1, 0, 0, 0), \tag{13}$$

and

$$\delta^{\mathrm{H}} = \mathrm{diag}(0, 0, 0, 1, 1, 1). \tag{14}$$

It is easily verified that the relations

$$\delta^{\mathrm{H}}\mathcal{S} = \mathcal{S}\delta^{\mathrm{E}}, \tag{15}$$

and

$$\delta^{\mathrm{E}}\mathcal{S} = \mathcal{S}\delta^{\mathrm{H}} \tag{16}$$

hold. These equations show that when matrix $\mathcal{S}$ operates on a vector related to the electric field strength, a vector related to the magnetic field strength results and vice versa.

3. BASIC FDTD EQUATIONS

In this section we briefly review the basic FDTD equations. Much more about the finite-difference discretization of Maxwell's equations can be found in the books by Taflove [3, 4] and the book by Kunz and Luebbers [2].

We introduce a uniform grid $\{x_{1;i}, x_{2;j}, x_{3;k}\} = \{(i-1)\Delta, (j-1)\Delta, (k-1)\Delta; i, j, k = 1, 2, \ldots, N_\mathrm{s}\}$, where $\Delta = 1/(N_\mathrm{s} - 1)$ is the grid size, and approximate the electromagnetic field quantities in a staggered manner. For example, the finite-difference approximation of the first component of the first Maxwell equation is

$$\begin{aligned}
&-\frac{H_3\left(x_{1;i+1/2}, x_{2;j+1/2}, x_{3;k}, t\right) - H_3\left(x_{1;i+1/2}, x_{2;j-1/2}, x_{3;k}, t\right)}{\Delta} \\
&+\frac{H_2\left(x_{1;i+1/2}, x_{2;j}, x_{3;k+1/2}, t\right) - H_2\left(x_{1;i+1/2}, x_{2;j}, x_{3;k-1/2}, t\right)}{\Delta} \\
&+\varepsilon_\mathrm{r}\left(x_{1;i+1/2}, x_{2;j}, x_{3;k}\right)\partial_t E_1\left(x_{1;i+1/2}, x_{2;j}, x_{3;k}, t\right) = -J_1^\mathrm{e}\left(x_{1;i+1/2}, x_{2;j}, x_{3;k}, t\right),
\end{aligned} \tag{17}$$

and similar expressions hold for the other equations. At the boundary of the computational domain we set the tangential electric field strength components to zero. For example, in the plane $x_2 = x_{2;1}$, the components E_1 and E_3 are set to zero. The resulting set of equations can be written compactly as

$$(D + M\partial_t)F(t) = Q(t). \tag{18}$$

This equation is similar in form to Eq. (8). The matrices D and M are both square and of order N; matrix D represents the spatial differentiation operator matrix $\mathcal{S}$ and for a uniform grid it is skew-symmetric, while matrix M represents the medium matrix $\mathcal{M}$ and is diagonal and positive definite. The counterparts of δ^E and δ^H are denoted by the same symbols. It is not difficult to verify that Eqs. (15) and (16) have an analog after discretization. Note that in two dimensions, N is proportional to $3N_\mathrm{s}^2$, while in three dimensions, N is proportional to $6N_\mathrm{s}^3$.

Multiply Eq. (18) on the left by δ^E. With the help of the discrete counterpart of Eq. (16) we obtain

$$D\delta^\mathrm{H}F(t) + M\partial_t\delta^\mathrm{E}F(t) = \delta^\mathrm{E}Q(t). \tag{19}$$

Introduce the time instances $t_n = n\Delta t$, where $\Delta t > 0$ is the time step, and n is an integer. Integrating Eq. (19) in time, with $t = t_n$ and $t = t_{n+1}$ as integration limits, leads to

$$D\int_{\tau=t_n}^{t_{n+1}} \delta^\mathrm{H}F(\tau)\,\mathrm{d}\tau + M[\delta^\mathrm{E}F(t_{n+1}) - \delta^\mathrm{E}F(t_n)] = I_n\{\delta^\mathrm{E}Q\}, \tag{20}$$

where

$$I_n\{\delta^\mathrm{E}Q\} = \int_{\tau=t_n}^{t_{n+1}} \delta^\mathrm{E}Q(\tau)\,\mathrm{d}\tau. \tag{21}$$

Using the midpoint rule to approximate the integral on the left-hand side of Eq. (20) results in

$$\Delta t\, D\,\delta^\mathrm{H}F\left(t_{n+1/2}\right) + M[\delta^\mathrm{E}F(t_{n+1}) - \delta^\mathrm{E}F(t_n)] = I_n\{\delta^\mathrm{E}Q\}. \tag{22}$$

If we solve this equation for the electric field strength at the time instant $t = t_{n+1}$, we obtain the first FDTD equation (see Taflove [3])

$$\delta^{\mathrm{E}} F(t_{n+1}) = \delta^{\mathrm{E}} F(t_n) - \Delta t M^{-1} D\, \delta^{\mathrm{H}} F\left(t_{n+1/2}\right) + M^{-1} I_n\{\delta^{\mathrm{E}} Q\}. \tag{23}$$

Notice that computing M^{-1} is trivial since matrix M is diagonal. Moreover, only the permittivity values need to be inverted since M^{-1} appears in a product with δ^{E} in the above equation. (Recall that $M^{-1} D \delta^{\mathrm{H}} = M^{-1} \delta^{\mathrm{E}} D$.)

Now multiply Eq. (18) on the left by δ^{H}. Using the discrete counterpart of Eq. (15) and integrating the resulting expression in time with $t = t_{n-1/2}$ and $t = t_{n+1/2}$ as integration limits give

$$D \int_{\tau=t_{n-1/2}}^{t_{n+1/2}} \delta^{\mathrm{E}} F(\tau)\, \mathrm{d}\tau + M\left[\delta^{\mathrm{H}} F\left(t_{n+1/2}\right) - \delta^{\mathrm{H}} F\left(t_{n-1/2}\right)\right] = I_{n-1/2}\{\delta^{\mathrm{H}} Q\}. \tag{24}$$

Again, we use the midpoint rule to approximate the integral on the left-hand side of Eq. (24). This gives

$$\Delta t D\, \delta^{\mathrm{E}} F(t_n) + M\left[\delta^{\mathrm{H}} F\left(t_{n+1/2}\right) - \delta^{\mathrm{H}} F\left(t_{n-1/2}\right)\right] = I_{n-1/2}\{\delta^{\mathrm{H}} Q\}. \tag{25}$$

Solving this equation for the magnetic field strength at $t = t_{n+1/2}$ gives the second time stepping equation of the FDTD scheme (see Taflove [3])

$$\delta^{\mathrm{H}} F\left(t_{n+1/2}\right) = \delta^{\mathrm{H}} F\left(t_{n-1/2}\right) - \Delta t M^{-1} D \delta^{\mathrm{E}} F(t_n) + M^{-1} I_{n-1/2}\{\delta^{\mathrm{H}} Q\}. \tag{26}$$

In this equation only the permeability values need to be inverted.

4. STABILITY ANALYSIS

Instead of solving for the electric and magnetic field components at the latest time instant, we write the basic FDTD equations in a different form. To this end we introduce the field vector $\tilde{\mathsf{F}}$ as

$$\tilde{\mathsf{F}}(t_n) = \begin{pmatrix} \delta^{\mathrm{E}} F(t_n) \\ \delta^{\mathrm{H}} F\left(t_{n-1/2}\right) \end{pmatrix}, \tag{27}$$

and write Eqs. (22) and (25) in the form

$$\begin{pmatrix} M & \Delta t D \\ 0 & M \end{pmatrix} \tilde{\mathsf{F}}(t_{n+1}) = \begin{pmatrix} M & 0 \\ -\Delta t D & M \end{pmatrix} \tilde{\mathsf{F}}(t_n) + \begin{pmatrix} I_n\{\delta^{\mathrm{E}} Q\} \\ I_{n-1/2}\{\delta^{\mathrm{H}} Q\} \end{pmatrix}. \tag{28}$$

Since matrix M is diagonal, we conclude from Eq. (28) that one FDTD step is equivalent to a forward–backward substitution.

To study the stability of the FDTD method, it is sufficient to consider the homogeneous counterpart of Eq. (28). Introduce the iteration matrix

$$G = \begin{pmatrix} I + (\Delta t A)^2 & -\Delta t A \\ -\Delta t A & I \end{pmatrix}, \tag{29}$$

where

$$A = M^{-1/2} D M^{-1/2}, \tag{30}$$

and rewrite the homogeneous system as

$$\begin{pmatrix} M^{1/2} & 0 \\ 0 & M^{1/2} \end{pmatrix} \tilde{\mathsf{F}}(t_{n+1}) = G \begin{pmatrix} M^{1/2} & 0 \\ 0 & M^{1/2} \end{pmatrix} \tilde{\mathsf{F}}(t_n). \tag{31}$$

From this equation it follows that

$$\mathsf{F}(t_{n+m}) = G^m \mathsf{F}(t_n), \tag{32}$$

where we have introduced the vector

$$\mathsf{F}(t_n) = \begin{pmatrix} M^{1/2} & 0 \\ 0 & M^{1/2} \end{pmatrix} \tilde{\mathsf{F}}(t_n). \tag{33}$$

We now give the following definition of stability of the FDTD method.

DEFINITION 4.1. The FDTD method is stable if for every $m \geq 1$ and fixed n there exists a constant K such that

$$\|\mathsf{F}(t_{n+m})\|_2 \leq K \|\mathsf{F}(t_n)\|_2. \tag{34}$$

The choice for the 2-norm poses no restrictions since all norms are equivalent on a finite-dimensional space. We have taken the 2-norm for convenience and because $\|\mathsf{F}\|_2^2$ is a measure for the stored electromagnetic energy in the computational domain.

Since matrix A is skew-symmetric, it can be diagonalized by a unitary similarity transformation; that is, there exists a matrix X such that

$$AX = X\Lambda, \quad \text{where } X^H X = X X^H = I, \tag{35}$$

and $\Lambda = \text{diag}(i\zeta_1, i\zeta_2, \ldots, i\zeta_N)$ with $i^2 = -1$ and $\zeta_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$. The spectral radius of matrix A is given by $\rho(A) = \max_j |\zeta_j|$. Notice that there are at least two eigenvalues whose absolute value is equal to the spectral radius of matrix A.

With the eigendecomposition of matrix A at our disposal, we can write

$$\begin{pmatrix} X^H & 0 \\ 0 & X^H \end{pmatrix} G \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} I + (\Delta t \Lambda)^2 & -\Delta t \Lambda \\ -\Delta t \Lambda & I \end{pmatrix}, \tag{36}$$

and the matrix on the right-hand side of the above equation can be permuted into the form $\text{diag}(B_1, B_2, \ldots, B_N)$, where each B_j is a 2-by-2 matrix given by

$$B_j = \begin{pmatrix} 1 - (\Delta t \zeta_j)^2 & -i\Delta t \zeta_j \\ -i\Delta t \zeta_j & 1 \end{pmatrix}. \tag{37}$$

To clarify this last step, consider the matrix

$$R = \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & \alpha_2 & 0 & \beta_2 \\ \beta_1 & 0 & 1 & 0 \\ 0 & \beta_2 & 0 & 1 \end{pmatrix}. \tag{38}$$

This matrix has the same structure as the matrix on the right-hand side of Eq. (36). If we interchange rows two and three and subsequently columns two and three, we obtain a matrix of the desired form. The interchanging of the relevant rows and columns is achieved by multiplying matrix R on the left and on the right by the permutation matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{39}$$

Note that matrix P is unitary and that the eigenvalues of PRP are the same as those of R. As is easily verified, matrix PRP is given by

$$PRP = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \beta_1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & \beta_2 & 1 \end{pmatrix}. \tag{40}$$

To summarize, there exists a unitary matrix U such that

$$U^H G U = B, \tag{41}$$

where matrix B is given by

$$B = \mathrm{diag}(B_1, B_2, \ldots, B_N), \tag{42}$$

and the 2-by-2 matrices B_j are given by Eq. (37). From this result it immediately follows that the $2N$ eigenvalues of the FDTD iteration matrix G are given by

$$\lambda_j^{\pm} = 1 - \frac{1}{2}(\Delta t \zeta_j)^2 \pm \frac{1}{2}\Delta t \zeta_j \sqrt{(\Delta t \zeta_j)^2 - 4} \quad \text{for } j = 1, 2, \ldots, N. \tag{43}$$

We are now in a position to prove our main result.

THEOREM 4.1. *A necessary condition for stability of the FDTD method is*

$$\Delta t \leq \frac{2}{\rho(A)}, \tag{44}$$

while a necessary and sufficient condition is

$$\Delta t < \frac{2}{\rho(A)}. \tag{45}$$

Proof. We prove the necessary condition first. If the FDTD scheme is stable we must have $\rho(G) \leq 1$. From Eq. (43) it then follow that

$$\Delta t \leq \frac{2}{\rho(A)}. \tag{46}$$

The proof of the sufficiency condition is more involved. If the condition of Eq. (45) is satisfied, all submatrices B_j can be diagonalized. In other words, there exist nonsingular matrices V_j such that

$$B_j = V_j \Sigma_j V_j^{-1}, \quad \text{for } j = 1, 2, \ldots, N, \tag{47}$$

where Σ_j is a 2-by-2 diagonal matrix with the eigenvalues of matrix B_j on its diagonal. From Eq. (43) it follows that all eigenvalues of matrix G (and hence of B) are located *on* the unit circle in the complex plane if the condition of Eq. (45) is satisfied.

We have

$$\|\mathsf{F}(t_{n+m})\|_2 = \|G^m \mathsf{F}(t_n)\|_2 \leq \|G^m\|_2 \|\mathsf{F}(t_n)\|_2, \tag{48}$$

and stability is proven by showing that $\|G^m\|_2$ remains bounded for any $m \geq 1$.

Now it is easily verified that

$$\|G^m\|_2 = \|B^m\|_2 = \max_{j=1,2,\ldots,N} \left\|B_j^m\right\|_2. \tag{49}$$

Say that it is the Jth submatrix for which the maximum is attained. Then,

$$\|G^m\|_2 = \left\|B_J^m\right\|_2 \leq \|V_J\|_2 \left\|\Sigma_J^m\right\|_2 \left\|V_J^{-1}\right\|_2 = \|V_J\|_2 \left\|V_J^{-1}\right\|_2 = K. \tag{50}$$

Equality in Eq. (46) is not a sufficient condition for stability. To show this, we construct a vector x for which $\|G^m x\|_2$ becomes unbounded as $m \to \infty$, if Δt is chosen such that $\Delta t \rho(A) = 2$. We have

$$\|G^m x\|_2^2 = \|U B^m U^H x\|_2^2 = \|B^m U^H x\|_2^2. \tag{51}$$

Now choose x as $x = Up$, and partition vector p as $p = [p_1^T, p_2^T, \ldots, p_N^T]^T$, where the p_j are 2-by-1 vectors. We then have

$$\|G^m x\|_2^2 = \sum_{j=1}^{N} \left\|B_j^m p_j\right\|_2^2. \tag{52}$$

Since $\Delta t \rho(A) = 2$, there are at least two submatrices equal to

$$\tilde{B} = \begin{pmatrix} -3 & -2i \\ -2i & 1 \end{pmatrix}. \tag{53}$$

Matrix $\tilde{B}$ cannot be diagonalized. Instead, we compute its Schur decomposition

$$\tilde{B} = W T W^H, \tag{54}$$

where

$$W = \frac{1}{2}\sqrt{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} -1 & -4i \\ 0 & -1 \end{pmatrix}. \tag{55}$$

Notice that matrix W is unitary and that the eigenvalues of $\tilde{B}$ are located on the diagonal of matrix T. From the expression for matrix T in Eq. (55) it follows that

$$T^m = (-1)^m \begin{pmatrix} 1 & 4mi \\ 0 & 1 \end{pmatrix}. \tag{56}$$

Let the Rth submatrix be equal to $\tilde{B}$. Choose vector p such that $p_j = [0\ \ 0]^T$ for $j \neq R$, and $p_R = W[0\ \ 1]$. With $x = Up$ and this particular vector p, we obtain

$$\begin{aligned} \|G^m x\|_2^2 &= \left\| B_R^m p_R \right\|_2^2 = \|W T^m W^H p_R\|_2^2 \\ &= \left\| T^m \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_2^2 = 16m^2 + 1 \to \infty \quad \text{as } m \to \infty, \end{aligned} \tag{57}$$

showing that there exits a vector x for which the scheme becomes unstable. ∎

The spectral radius of matrix A does not change if the mesh size is fixed and stable results are obtained on any given time interval as long as Eq. (45) is satisfied. However, the spectral radius increases for a finer mesh. Equation (45) tells us that a smaller time step is then necessary to obtain stable results.

As an interesting by-product, we show the 2-norm of the FDTD iteration matrix. Direct computation yields

$$\|G\|_2 = \sqrt{\frac{2 + s^2(s^2 + \sqrt{4 + s^4})}{2}}, \tag{58}$$

where $s = \Delta t \rho(A)$. This results shows that the 2-norm of the FDTD matrix is greater than one for any (positive) value of s. For $s = 1$ (Δt equals half the upper limit of Eq. (45)) we have

$$\|G\|_2 = \frac{1 + \sqrt{5}}{2}, \tag{59}$$

the golden ratio.

5. THE STABILITY CONDITION IN PRACTICE

The problem with the condition of Eq. (45) is that the spectral radius of matrix A is usually not available. In contrast, the infinity norm of this matrix is easily computed. Approximating the spectral radius by the infinity norm leads to the practical stability condition

$$\Delta t < \frac{2}{\|A\|_\infty}. \tag{60}$$

If the above inequality is satisfied, then the stability condition of Eq. (45) is automatically satisfied since $\rho(A) \leq \|A\|_\infty$.

As an illustration, consider a homogeneous medium characterized by a constant relative permittivity ε_{r} and a constant relative permeability μ_{r}. Using the definition of matrix A (see Eq. (30)), we obtain

$$\|A\|_\infty = \max_{i=1,2,\ldots,N} \sum_{j=1}^{N} |a_{i,j}| \\ = \begin{cases} \frac{2}{\sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}\Delta}, & \text{for one dimensional problems,} \\ \frac{4}{\sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}\Delta}, & \text{for two- and three-dimensional problems.} \end{cases} \tag{61}$$

Substituting this result in Eq. (60) then leads to the stability condition

$$\Delta t < f \cdot \sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}\Delta, \tag{62}$$

where $f = 1$ for one-dimensional problems, and $f = 1/2$ for two- and three-dimensional problems. Rewriting the above equation in terms of the unnormalized quantities gives

$$\Delta t < f \frac{\Delta}{c}, \tag{63}$$

where c is the electromagnetic wave speed.

6. A ONE-DIMENSIONAL CONFIGURATION

Consider the electromagnetic field in a configuration that is invariant in the x_1 and x_2 directions and let the normalized field quantities satisfy the equations

$$\partial_3 H_2 + \varepsilon_{\mathrm{r}}\partial_t E_1 = -J_1^{\mathrm{e}} \tag{64}$$

and

$$\partial_3 E_1 + \mu_{\mathrm{r}}\partial_t H_2 = -K_2^{\mathrm{e}}, \tag{65}$$

with $0 < x_3 < 1$. The system is initially at rest. The boundary conditions are given by

$$E_1(0,t) = E_1(1,t) = 0, \quad \text{for } t > 0. \tag{66}$$

Define the grid points $x_{3;i} = (i-1)\Delta$ for $i = 1, 2, \ldots, N_{\mathrm{s}}$ with $\Delta = 1/(N_{\mathrm{s}}-1)$, and approximate the electromagnetic field quantities on a staggered grid (see Fig. 1). We then obtain the equations

$$\frac{H_2\left(x_{3;i+1/2},t\right) - H_2\left(x_{3;i-1/2},t\right)}{\Delta} + \varepsilon_{\mathrm{r}}(x_{3;i})\partial_t E_1(x_{3;i},t) = -J_1^{\mathrm{e}}(x_{3;i},t), \tag{67}$$

for $i = 2, 3, \ldots, N_{\mathrm{s}}-1$, and

$$\frac{E_1(x_{3;i+1},t) - E_1(x_{3;i},t)}{\Delta} + \mu_{\mathrm{r}}\left(x_{3;i+1/2}\right)\partial_t H_2\left(x_{3;i+1/2},t\right) = -K_2^{\mathrm{e}}\left(x_{3;i+1/2},t\right), \tag{68}$$

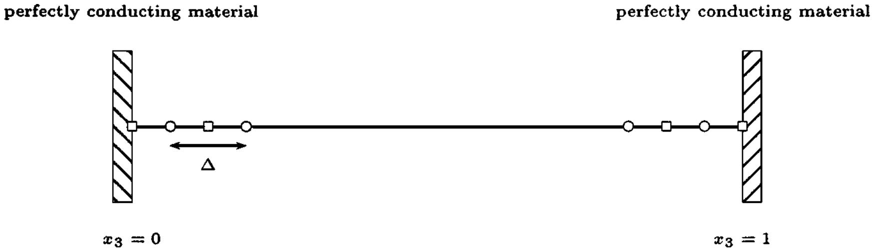

FIG. 1. A one-dimensional configuration. The squares indicate the location of the electric field strength component E_1, the circles indicate the location of the magnetic field strength component H_2.

for $i = 1, 2, \ldots, N_s - 1$. The boundary conditions become

$$E_1(x_{3;1}, t) = E_1\left(x_{3;N_s}, t\right) = 0, \quad \text{for } t > 0. \tag{69}$$

Equations (67)–(69) can be written more compactly as

$$(D + M\partial_t)F(t) = Q(t), \tag{70}$$

in which the field vector $F = F(t)$ is given by

$$F(t) = \left[H_2\left(x_{3;3/2}, t\right), E_1(x_{3;2}, t), H_2\left(x_{3;5/2}, t\right), \ldots, E_1\left(x_{3;N_s-1}, t\right), H_2\left(x_{3;N_s-1/2}, t\right)\right]^T, \tag{71}$$

and the source vector has a similar partitioning. The medium matrix M is of the form

$$M = \text{diag}\left(\mu_r\left(x_{3;3/2}\right), \varepsilon_r(x_{3;2}), \mu_r\left(x_{3;5/2}\right), \ldots, \varepsilon_r\left(x_{3;N_s-1}\right), \mu_r\left(x_{3;N_s-1/2}\right)\right), \tag{72}$$

and matrix D is given by

$$D = \text{tridiag}\left(-\frac{1}{\Delta}, 0, \frac{1}{\Delta}\right). \tag{73}$$

To simplify the analysis, we consider a homogeneous medium characterized by a constant relative permittivity ε_r and a constant relative permeability μ_r. As is easily verified, matrix A, as defined in Eq. (30), is then given by

$$A = \text{tridiag}\left(-\frac{1}{\sqrt{\varepsilon_r\mu_r}\Delta}, 0, \frac{1}{\sqrt{\varepsilon_r\mu_r}\Delta}\right). \tag{74}$$

For this particular example the spectral radius of matrix A is known. We have

$$\rho(A) = \frac{2}{\sqrt{\varepsilon_r\mu_r}\Delta} \cos\left(\frac{\pi\Delta}{2}\right). \tag{75}$$

Substitution of this result in Eq. (45) leads to the stability condition

$$\Delta t < \frac{\sqrt{\varepsilon_r\mu_r}\Delta}{\cos\left(\frac{\pi\Delta}{2}\right)}. \tag{76}$$

If we rewrite this inequality in terms of the unnormalized quantities, we obtain the bound

$$\Delta t < \frac{\Delta}{c\cos\left(\frac{\pi\Delta}{2L}\right)}. \tag{77}$$

In Figs. 2a–2c we have plotted the eigenvalues of the FDTD iteration matrix for a fixed N_s and three different values of Δt, namely, $\Delta t = 5/(2\rho(A))$, $\Delta t = 2/\rho(A)$, and

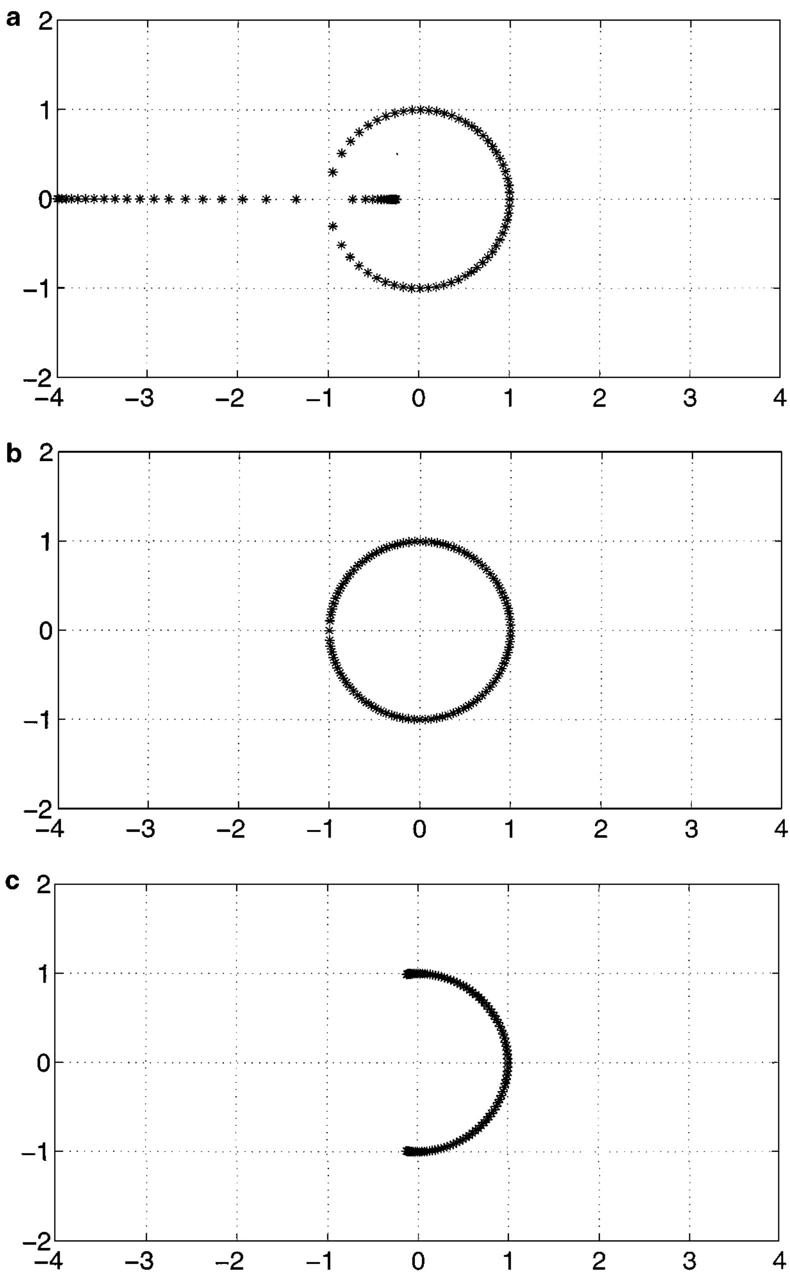

FIG. 2. (a) Eigenvalues of the FDTD iteration matrix G for $\Delta t = 5/(2\rho(A))$. (b) Eigenvalues of the FDTD iteration matrix G for $\Delta t = 2/\rho(A)$. (c) Eigenvalues of the FDTD iteration matrix G for $\Delta t = 3/(2\rho(A))$.

$\Delta t = 3/(2\rho(A))$. Notice that the eigenvalues approach one and remain on the unit circle as Δt becomes smaller. This can also be seen from Eq. (43), of course.

7. CONCLUSIONS

In this paper we have presented a necessary and sufficient condition for stability of the FDTD method. Furthermore, we have shown that for a particular time step the 2-norm of the FDTD iteration matrix is equal to the golden ratio. In our analysis we considered inhomogeneous, isotropic, and lossless media. Future work will focus on the incorporation of lossy media. Finally, we mention that the stability analysis presented in this paper can also be used to study the stability of FDTD schemes for acoustic and elastodynamic wave fields, since it can be shown that the governing equations for these types of waves can be written in the form of Eq. (8) as well (see De Hoop and De Hoop [1]).

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